HEIGHT IN SPLITTING OF RELATIVELY HYPERBOLIC GROUPS

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Abstract. Given a finite graph of relatively hyperbolic groups with its fundamental group relatively hyperbolic and edge groups embed quasi-isometrically in vertex groups. We prove that a vertex group is relatively quasiconvex if and only if all the vertex groups have finite height in the fundamental group.

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1. Introduction

A subgroup H of a group G is said to be malnormal if for all $g \in G \setminus H$, $H \cap gHg^{-1}$ is trivial. This concept was generalized to width and height of subgroups by Gitik-Mitra-Rips-Sageev in [4]. A subgroup H of a hyperbolic group G has height (resp. width) n, if there exists n distinct cosets $g_1H, ..., g_nH$ such that $\cap_{1\leq i\leq n} g_iHg_i{}^{-1}$ (resp. $g_iHg_i^{-1} \cap g_jHg_j^{-1}$ for $i \neq j$) is infinite and n is maximal possible. In [4], it has been proved that a quasiconvex subgroup of a hyperbolic group has finite height and width. Converse of this is still an open problem. As a partial converse, in [11], Mitra proved the following theorem:

Theorem 1.1. *(Theorem 4.6 of [11]) Suppose G is a hyperbolic group which splits over a subgroup H (i.e. $G = G_1 *_H G_2$ or $G = G_1 *_H$) with vertex groups and edge groups hyperbolic and inclusions of H in G_1, G_2 are quasi-isometric embeddings. Then, H has finite height if and only if H is quasiconvex in G.*

The notions of width and height for subgroups of relatively hyperbolic group were defined by Hruska and Wise in [7]. A subgroup H of a relatively hyperbolic group G has height (resp. width) n, if there exists n distinct cosets $g_1H, ..., g_nH$ such that $\cap_{1\leq i\leq n} g_iHg_i{}^{-1}$ (resp. $g_iHg_i^{-1} \cap g_jHg_j^{-1}$ for $i \neq j$) contains hyperbolic elements and n is maximal possible. In [7], Hruska and Wise proved that if a relatively quasiconvex subgroup H of a relatively hyperbolic group G has finite height and further if H satisfies a "bounded packing condition" with respect to the peripheral subgroups then H has finite width. As in case of hyperbolic groups, it is interesting to ask which subgroups of relatively hyperbolic groups have finite height or width? (Refer to problem list in Section 4.5 of [12]). In this article, we prove the following theorem:

Theorem 1.2. *(Theorem 4.1) Let $\delta \geq 0, k \geq 1, \epsilon \geq 0$. Let $(\mathcal{G}, \Lambda)$ be a finite graph of groups with its fundamental group $G = \pi_1(\mathcal{G}, \Lambda)$ finitely generated and torsion free. Consider the graph of subgroups $(\mathcal{H}, \Lambda)$ of $(\mathcal{G}, \Lambda)$ with $H = \pi_1(\mathcal{H}, \Lambda)$ and suppose the following conditions hold:*
(i) for every edge e incident on a vertex v of the graph Λ, the embeddings $i_e : G_e \hookrightarrow G_v$, $\widehat{i}_e : \mathcal{E}(G_e, H_e) \hookrightarrow \mathcal{E}(G_v, H_v)$ are (k, ϵ)-quasi-isometric, where $H_e = H_v \cap G_e$ and $\mathcal{E}(G_e, H_e), \mathcal{E}(G_v, H_v)$ denote coned-off graphs
(ii) G is δ-hyperbolic relative to H,
(iii) for each vertex v of Λ; G_v is δ-hyperbolic relative to H_v and G_v has finite height in G.
Then, for all vertex w, G_w is relatively quasiconvex subgroup of G.

Let $k \geq 1$ be a natural number. An action of a group G on a simplicial tree T without inversions is said to be *k-acylindrical* if no non-trivial element of G fixes pointwise a segment of length k in T. Let Λ be a non-empty connected graph. A graph of group $(\mathcal{G}, \Lambda)$ is called k-acylindrical (see [9]) if the action of the fundamental group of $(\mathcal{G}, \Lambda)$ (in the sense of Bass-Serre theory) on the Bass-Serre covering tree $\widetilde{\Lambda}$ of Λ is k-acylindrical. For an acylindrical HNN-extension $G = H*_A$, G acts on the Bass-Serre tree T with vertex groups conjugates of H and edge groups conjugates of

A. It can be shown that the subgroup H has finite height in G (refer to Page 3 of [9]). Thus, we have the following corollary:

Corollary 1.3. *Let H be a group and A, A_1, B, B_1 be subgroups of H such that $A_1 \subset A$ and $B_1 \subset B$ Let $\phi : A \to B$ be an isomorphism such that $\phi(A_1) = B_1$, $G =< H, t | tat^{-1} = \phi(a)\ \forall a \in A >$ be HNN extension of A with respect to ϕ and $G_1 =< A_1, t >$. Suppose both H and A are hyperbolic relative to subgroup A_1 and G is hyperbolic relative to G_1. Assume that the embeddings $A \hookrightarrow G$, $\mathcal{E}(A, A_1) \hookrightarrow \mathcal{E}(G, G_1)$ are quasi-isometric embeddings. If G is acylindrical then H is a relatively quasiconvex subgroup of G.*

In section 2, we first go through different definitions of relatively hyperbolic spaces. Then we give the notions of partial electrocution, tree of spaces, hyperbolic ladder which are main ingredients to prove the Theorem 4.1. In section 3, we define hallway and give the existence of hallway in the tree of relatively hyperbolic spaces. Finally, in section 4, we prove the main theorem.

2. Relatively Hyperbolic Groups

2.1. **Relatively Hyperbolic Spaces.** Let (X, d_X) be a path metric space. A collection of closed subsets $\mathcal{H} = \{H_\alpha\}_{\alpha\in\Lambda}$ of X will be said to be **uniformly separated** if there exists $\mu > 0$ such that $d_X(H_\alpha, H_\beta) \geq \mu$ for all distinct $H_\alpha, H_\beta \in \mathcal{H}$.

Let $Z = X \bigsqcup(\sqcup_\alpha(H_\alpha \times [0, \frac{1}{2}]))$ and we define a distance function on Z as follows:

$$\begin{aligned} d_Z(x, y) &= d_X(x, y), \text{ if } x, y \in X, \\ &= d_{H_\alpha\times[0,\frac{1}{2}]}(x, y), \text{ if } x, y \in H_\alpha \text{ for some } \alpha \in \Lambda, \\ &= \infty, \text{ if } x, y \text{ does not lie on a same set of the disjoint union.} \end{aligned}$$

The metric $d_{H_\alpha\times[0,\frac{1}{2}]}$ above is the product metric on $H_\alpha \times [0, \frac{1}{2}]$. Let $\{v_\alpha\}_{\alpha\in\Lambda}$ be a collection of distinct points indexed by Λ. Let $\sim$ be a relation on Z defined as follows:
$(h_\alpha, \frac{1}{2}) \sim v_\alpha$ and $(h_\alpha, 0) \sim h_\alpha$ for all $h_\alpha \in H_\alpha$ and for all $\alpha \in \Lambda$.
$\sim$ is an equivalence relation.

Definition 2.1. *(Farb [3]) The* **electric space** *(or coned-off space) $\mathcal{E}(X, \mathcal{H})$ corresponding to the pair $(X, \mathcal{H})$ is the quotient space $Z/\sim$ of Z and the points v_α are said to be cone points. We define a metric $d_{\mathcal{E}(X,\mathcal{H})}$ on $\mathcal{E}(X, \mathcal{H})$ as follows:*

$$d_{\mathcal{E}(X,\mathcal{H})}([x], [y]) = \inf \sum_{1\leq i\leq n} d_Z(x_i, y_i),$$

where the infimum is taken over all sequences $C = \{x_1, y_1, x_2, y_2, ..., x_n, y_n\}$ of points of Z such that $x_1 \in [x], y_n \in [y]$ and $y_i \sim x_{i+1}$ for $i = 1, ..., n - 1$. ($\sim$ is the equivalence relation on Z).

Note that $\mathcal{E}(X, \mathcal{H})$ is a metric space obtained from X by coning each H_α to the cone point v_α.

Definition 2.2. *X is said to be* **weakly δ-hyperbolic** *relative to the collection $\mathcal{H}$ if $\mathcal{E}(X, \mathcal{H})$ is a δ-hyperbolic metric space. Let G be a finitely generated group with generating set S and $X = \Gamma(G; S)$ be its Cayley graph. Let H be a finitely generated subgroup of G and $\mathcal{H}$ denotes the collection of left cosets of H in G. G is said to be weakly δ-hyperbolic relative to H if X is weakly δ-hyperbolic relative to $\mathcal{H}$. For the pair (G, H), we denote the coned-off graph $\mathcal{E}(X, \mathcal{H})$ by $\mathcal{E}(G, H)$.*

For a path $\gamma : [0, 1] \to X$, there is an induced path $\widehat{\gamma}$ in $\mathcal{E}(X, \mathcal{H})$ obtained as follows:
for each component $\gamma|_{[s,t]}$ of γ lying in a set $H_\alpha \in \mathcal{H}$, replace the component $\gamma|_{[s,t]}$ by the path $[\gamma(s), v_\alpha] \cup [v_\alpha, \gamma(t)]$ of length one passing through cone point v_α, where $[\gamma(s, v_\alpha)], [v_\alpha, \gamma(t)]$ are geodesics in the cone $\mathcal{E}(H_\alpha, \{H_\alpha\})$.
If $\widehat{\gamma}$ is a geodesic (resp. P-quasigeodesic) in $\mathcal{E}(X, \mathcal{H})$, γ is called a *relative geodesic* (resp. *relative P-quasigeodesic*). γ is said to be **without backtracking** if γ does not return to an H_α after leaving it.

Definition 2.3. [3] *Relative geodesics (resp. P-quasigeodesics) in $(X, \mathcal{H})$ are said to satisfy* **bounded penetration properties** *if there exists $K = K(P) > 0$ such that for any two relative geodesics (resp. P-quasigeodesics without backtracking) β, γ joining $x, y \in X$ following two*

properties are satisfied:
(1) if precisely one of $\{\beta, \gamma\}$ *meets a set* $H_\alpha \in \mathcal{H}$*, then the distance (measured in the intrinsic path-metric on* H_α*) from the first (entry) point to the last (exit) point (of the relevant path) is at most* K*,*
(2) if both $\{\beta, \gamma\}$ *meet some* $H_\alpha \in \mathcal{H}$ *then the distance (measured in the intrinsic path-metric on* H_α*) from the entry point of* β *to that of* γ *is at most* K*; similarly for exit points.*

Definition 2.4. *(Farb [3]) X is said to be* ***δ-hyperbolic relative*** *to the uniformly separated collection $\mathcal{H}$ if X is weakly δ-hyperbolic relative to $\mathcal{H}$ and relative P quasigeodesics without backtracking satisfy the bounded penetration properties. A finitely generated group G with generating set S is said to be δ-hyperbolic relative to a subgroup H if its Cayley graph $X = \Gamma(G, S)$ is δ-hyperbolic relative to $\mathcal{H} = \{aH : a \in G\}$*

The following proposition will be useful in proving Remark 3.5.

Proposition 2.5. *Let (Z, d) be a hyperbolic metric space which is hyperbolic relative to the collection of μ-separated closed sets $\mathcal{H}$, where $\mu > 1$.*
(i) For $c \geq 0$, if Q is a c-quasiconvex set in (Z, d) then there exists $r = r(c) \geq 0$ such that Q is r-quasiconvex in the coned-off space $\widehat{Z} = \mathcal{E}(Z, \mathcal{H})$.
(ii) Let
- *$\widehat{\lambda}$ be a geodesic in $\widehat{Z}$ joining two points $p, q \in Z$ and $\lambda^b = \widehat{\lambda} \setminus (\cup_{H \in \mathcal{H}} H)$,*
- *$N(\lambda)$ be the union of λ^b and elements of $\mathcal{H}$ which are penetrated by $\widehat{\lambda}$.*

Then $N(\lambda)$ is a quasiconvex set in Z

Proof. (i) Obvious as $d_{\widehat{Z}}(x, y) \leq max\{d_Z(x, y), 1\}$.
(ii) Let β be a hyperbolic geodesic in Z joining two points of $N(\lambda)$. Due to bounded penetration properties (2.3), outside the elements of $\mathcal{H}$, β and $\widehat{\lambda}$ track each other i.e. they lie in bounded neighborhood of each other in Z. Thus, $N(\lambda)$ is quasiconvex in Z. ☐

2.2. Equivalent definitions of Relatively Hyperbolic Spaces.

Definition 2.6. *(Hyperbolic Cones:) For any geodesic metric space (H, d), the* ***hyperbolic cone*** *(analog of a horoball), denoted by H^h, is topologically the space $H \times [0, \infty)$ equipped with the metric d_{H^h} defined as follows:*
Let $\alpha : [0, 1] \to H \times [0, \infty) = H^h$ be a path then $\alpha = (\alpha_1, \alpha_2)$, where α_1, α_2 are coordinate functions. Suppose $\tau :\ 0 = t_0 < t_1 < ... < t_n = 1$ be a partition of $[0, 1]$. Define the length of α by

$$l_{H^h}(\alpha) = \lim_{\tau} \sum_{1 \leq i \leq n-1} \sqrt{e^{-2\alpha_2(t_i)} d_H(\alpha_1(t_i), \alpha_1(t_{i+1}))^2 + |\alpha_2(t_i) - \alpha_2(t_{i+1})|^2},$$

Here the limit is taken with respect to the refinement ordering of partitions over $[0, 1]$. The distance between two points $x, y \in H^h$ is defined to be

$$d_{H^h}(x, y) = inf\{l_{H^h}(\alpha) : \alpha \text{ is a curve from } x \text{ to } y\}.$$

Proposition 2.7. [2] *For any geodesic metric space (H, d), there exists $\triangle \geq 0$ such that the hyperbolic cone (H^h, d_{H^h}) is a $\triangle$-hyperbolic metric space.*

The combinatorial description of hyperbolic cone was given by Groves and Manning in [6] (Refer to section 3 of [6]).
Gromov's definition of relative hyperbolicity [5] :
Let (X, d_X) be geodesic metric space and $\mathcal{H} = \{H_\alpha : \alpha \in \Lambda\}$ be a collection of uniformly ν-separated, intrinsically geodesic, closed subsets of X.
Let $Z = X \bigsqcup (\sqcup_{\alpha \in \Lambda} H_\alpha^h)$. Define a distance function d_Z on Z as follows:

$$\begin{aligned} d_Z(x, y) &= d_X(x, y), \text{ if } x, y \in X, \\ &= d_{H_\alpha^h}(x, y), \text{ if } x, y \in H_\alpha \text{ for some } \alpha \in \Lambda, \\ &= \infty, \text{ if } x, y \text{ does not lie on a same set of the disjoint union.} \end{aligned}$$

Let $\mathcal{G}(X,\mathcal{H})$ be the quotient space of Z obtained by attaching the hyperbolic cones H^h_α to $H_\alpha \in \mathcal{H}$ by identifying $(z,0)$ with z, for all $H_\alpha \in \mathcal{H}$ and $z \in H_\alpha$.
We define a metric $d_{\mathcal{G}(X,\mathcal{H})}$ on $\mathcal{G}(X,\mathcal{H})$ as follows:

$$d_{\mathcal{G}(X,\mathcal{H})}([x],[y]) = \inf \sum_{1\leq i\leq n} d_Z(x_i,y_i),$$

where the infimum is taken over all sequences $C = \{x_1, y_1, x_2, y_2, ..., x_n, y_n\}$ of points of Z such that $x_1 \in [x], y_n \in [y]$ and $y_i \sim x_{i+1}$ for $i = 1, ..., n-1$. ($\sim$ is the equivalence relation on Z). In short, $(\mathcal{G}(X,\mathcal{H}), d_{\mathcal{G}(X,\mathcal{H})})$ will be denoted by (X^h, d_{X^h}).

Definition 2.8. *Let $\delta \geq 0, \nu > 1$. Let X be a geodesic metric space and $\mathcal{H}$ be a collection of uniformly ν-separated, intrinsically geodesic closed subsets of X. X is said to be* ***δ-hyperbolic relative to $\mathcal{H}$ in the sense of Gromov*** *if the space X^h is a δ-hyperbolic metric space. X is said to be hyperbolic relative to $\mathcal{H}$ in the sense of Gromov if X is δ-hyperbolic relative to $\mathcal{H}$ in the sense of Gromov for some $\delta \geq 0$.*

Definition 2.9. *(Relatively Quasiconvex Subgroup): Let $q \geq 0$. Suppose G is a finitely generated group hyperbolic relative to a collection $\{H_1, ..., H_m\}$ of finitely generated subgroups of G. A subgroup K of G is said to be q-relatively quasiconvex if K is a q-quasiconvex set in $\widehat{\Gamma}_G$.*

Next we describe a special type of quasigeodesic, called electro-ambient quasigeodesic, which will be useful to construct hyperbolic ladder in section 2.5.

Definition 2.10. *Let X be hyperbolic relative to $\mathcal{H}$. We start with an electric quasi-geodesic $\hat{\lambda}$ in the electric space $\mathcal{E}(X,\mathcal{H})$ without backtracking. For any H penetrated by $\hat{\lambda}$, let x_H and y_H be the first entry point and the last exit point of $\hat{\lambda}$. We join x_H and y_H by a hyperbolic geodesic segment in H^h (identifying $\mathcal{E}(X,\mathcal{H})$ with $\mathcal{E}(\mathcal{G}(X,\mathcal{H}),\mathcal{H}^h)$). This results in a path λ_h in $\mathcal{G}(X,\mathcal{H})$. The path λ_h will be called an* **electro-ambient quasigeodesic**.

Lemma 2.11. *An electro-ambient quasigeodesic is a quasigeodesic in $\mathcal{G}(X,\mathcal{H})$.*

2.3. **Partial Electrocution.** The notion of Partial Electrocution was introduced by Mahan Mj. in [12]. This is a modification of Farb's [3] construction of an electric space described earlier. In a partially electrocuted space, instead of coning all of a horosphere down to a point we cone only it to a hyperbolic metric space.

Definition 2.12. [12] *Let $\delta \geq 0, \nu > 0$ and $(X,\mathcal{H},\mathcal{G},\mathcal{L})$ be an ordered quadruple such that the following holds:*

1. *X is a geodesic metric space and $\mathcal{H} = \{H_\alpha : \alpha \in \Lambda\}$ is a collection of uniformly ν-separated, intrinsically geodesic and uniformly properly embedded closed subsets of X. X is δ-hyperbolic relative to $\mathcal{H}$ in the sense of Gromov.*
2. *$\mathcal{L} = \{L_\alpha : \alpha \in \Lambda\}$ is a collection of δ-hyperbolic geodesic metric spaces and $\mathcal{G}$ is a collection of (uniformly) Lipschitz onto maps $g_\alpha : H_\alpha \to L_\alpha$ i.e. there exists a number $P > 0$ such that $d_{L_\alpha}(g_\alpha(x), g_\alpha(y)) \leq P d_{H_\alpha}(x,y)$ for all $x, y \in H_\alpha$ and for all index α. Note that the indexing set for $H_\alpha, L_\alpha, g_\alpha$ is common.*

The **partially electrocuted space** *or* partially coned off space *$\mathcal{PE}(X,\mathcal{H},\mathcal{G},\mathcal{L})$ corresponding to $(X,\mathcal{H},\mathcal{G},\mathcal{L})$ is the quotient space obtained from X as follows:*
$\mathcal{PE}(X,\mathcal{H},\mathcal{G},\mathcal{L}) = X \sqcup (\sqcup_\alpha (H_\alpha \times [0,1])) \sqcup (\sqcup_\alpha L_\alpha)/\cup_\alpha \{(x,0) \sim x, (x,1) \sim g_\alpha(x) : x \in H_\alpha\}$. (The metric on $H_\alpha \times [0,1]$ is the product metric.)
$\mathcal{PE}(X,\mathcal{H},\mathcal{G},\mathcal{L})$ is equipped with the quotient metric and the metric is denoted by d_{pel}. In short, $\mathcal{PE}(X,\mathcal{H},\mathcal{G},\mathcal{L})$ will be denoted by X_{pel}.

Lemma 2.13. *(Lemma 2.8 of [12]) (X, d_{pel}) is a hyperbolic metric space and the sets L_α are uniformly quasiconvex.*

Lemma 2.14. *(Lemma 2.9 of [12]) Given $K, \epsilon \geq 0$, there exists $C > 0$ such that the following holds:*

Let γ_{pel} and γ denote respectively a (K,ϵ) partially electrocuted quasigeodesic in (X, d_{pel}) and a hyperbolic (K,ϵ)-quasigeodesic in (X^h, d_{X^h}) joining a, b. Then $\gamma \cap X$ lies in a (hyperbolic) C-neighborhood of (any representative of) γ_{pel}. Further, outside of a C-neighborhood of the horoballs that γ meets, γ and γ_{pel} track each other.

2.4. **Trees of Spaces.**

Definition 2.15. *(Bestvina-Feighn [1]) Let $K \geq 1, \epsilon \geq 0$. $P : X \to T$ is said to be a tree of geodesic metric spaces satisfying the (K,ϵ)-q(uasi) i(sometrically) embedded condition if the geodesic metric space (X, d_X) admits a map $P : X \to T$ onto a simplicial tree T, such that there exist ϵ and $K > 0$ satisfying the following:*
1) For all $s \in T$, $X_s = P^{-1}(s) \subset X$ with the induced path metric d_{X_s} is a geodesic metric space X_s. Further, the inclusions $i_s : X_s \to X$ are uniformly proper, i.e. for all $M > 0$ there exists $N > 0$ such that for all $s \in T$ and $x, y \in X_s$, $d_X(i_s(x), i_s(y)) \leq M$ implies $d_{X_s}(x, y) \leq N$.
2) For a vertex v in T, $X_v = P^{-1}(v)$ will be called as vertex space for v. Let e be an edge of T with initial and final vertices v_1 and v_2 respectively. Let X_e be the pre-image under P of the mid-point of e, X_e will be called as edge space for e. There exist continuous maps $f_e : X_e \times [0,1] \to X$, such that $f_e|_{X_e \times (0,1)}$ is an isometry onto the pre-image of the interior of e equipped with the path metric. Further, f_e is fiber-preserving, i.e. projection to the second co-ordinate in $X_e \times [0,1]$ corresponds via f_e to projection to the tree $P : X \to T$.
3) Let v_1, v_2 be end points of e. $f_e|_{X_e \times \{0\}}$ and $f_e|_{X_e \times \{1\}}$ are (K,ϵ)-quasi-isometric embeddings into X_{v_1} and X_{v_2} respectively. $f_e|_{X_e \times \{0\}}$ and $f_e|_{X_e \times \{1\}}$ will occasionally be referred to as f_{e,v_1} and f_{e,v_2} respectively.

Let $\delta \geq 0$. A tree of spaces as in Definition 2.15 above is said to be a tree of δ-hyperbolic metric spaces, if X_v, X_e are all δ-hyperbolic for all vertices v and edges e of T.
• Define $\phi_{v,e} \colon f_{e,v_-}(X_e) \to f_{e,v}(X_e)$ as follows:
If $p \in f_{e,v_-}(X_e) \subset X_{v_-}$, choose $x \in X_e$ such that $p = f_{e,v_-}(x)$ and define $\phi_{v,e}(p) = f_{e,v}(x)$.
Note that in the above definition, x is chosen from a set of bounded diameter.

Since $f_{e,v_-}|_{X_e}$ and $f_{e,v}|_{X_e}$ are quasi-isometric embeddings into their respective vertex spaces $\phi_{v,e}$'s are uniform quasi-isometries for all vertices.

Now, let S be a hyperbolic once punctured surface with finite volume. $\pi_1(S)$ acts properly discontinuously on $\mathbb{H}^2$ and $\widetilde{S} = \mathbb{H}^2$. Let N denote S minus cusps and $\mathcal{B}$ be the collection of horodisks in $\mathbb{H}^2$ such that each element B of $\mathcal{B}$ projects down to the cusp under the quotient map $q : \mathbb{H}^2 \to \mathbb{H}^2/\pi_1(S)$, then $\widetilde{N}$ is equal to $\mathbb{H}^2$ minus horodisks in $\mathcal{B}$. $\pi_1(S)$ acts properly discontinuously and cocompactly on $\widetilde{N}$, therefore $\widetilde{N}$ is quasi-isometric to the Cayley graph of $\pi_1(S)$. Let $\phi : S \to S$ be an orientation preserving homeomorphism fixing the puncture. Then $\widetilde{\phi} : \widetilde{S} \to \widetilde{S}$ preserves corresponding horodisks. Therefore $\widetilde{\phi}$ induces a (K,ϵ) quasi-isometry $\widetilde{\Phi} : \widetilde{N} \to \widetilde{N}$. Let $N = \widetilde{N}/\pi_1(S)$ and $N_\phi = \frac{N \times [0,1]}{\{(x,0),(\phi(x),1) : x \in N\}}$. Then $\widetilde{N}_\phi$ can be treated as a tree of spaces with vertex spaces and edge spaces homeomorphic to $\widetilde{N}$.

Definition 2.16. *Let $\delta \geq 0, \nu \geq 1, \widehat{K} \geq 1, \widehat{\epsilon} \geq 0$ and X be a geodesic space. A tree $P : X \to T$ of geodesic metric spaces is said to be a tree of δ-relatively hyperbolic metric spaces if in addition to above three conditions of Definition 2.15, we have the following conditions:*
4) for each vertex space X_v (resp. edge space X_e) there exists a collection $\mathcal{H}_v$ (resp. $\mathcal{H}_e$) of uniformly ν-separated, intrinsically geodesic and uniformly properly embedded closed subsets of X_v (resp. X_e) such that X_v (resp. X_e) is δ-hyperbolic relative to the collection $\mathcal{H}_v$ (resp. $\mathcal{H}_e$) in the sense of Gromov,
5) the maps f_{e,v_i} above $(i = 1,2)$ are **strictly type-preserving**, *i.e. $f^{-1}_{e,v_i}(H_{\alpha v_i})$, $i = 1,2$ (for any $H_{\alpha v_i} \in \mathcal{H}_{v_i}$) is either empty or some $H_{\beta e} \in \mathcal{H}_e$. Also, for all $H_{\beta e} \in \mathcal{H}_e$, there exists v and $H_{\alpha v}$, such that $f_{e,v}(H_{\beta e}) \subset H_{\alpha v}$, and*
6) the induced maps (see below) of the coned-off edge spaces into the coned-off vertex spaces $\widehat{f_{e,v_i}}$: $\mathcal{E}(X_e, \mathcal{H}_e) \to \mathcal{E}(X_{v_i}, \mathcal{H}_{v_i})$ $(i = 1,2)$ are uniform $(\widehat{K},\widehat{\epsilon})$-quasi-isometric embeddings. This is called the **qi-preserving electrocution condition**

We shall denote $\mathcal{E}(X_v, \mathcal{H}_v) = \widehat{X_v}$ and $\mathcal{E}(X_e, \mathcal{H}_e) = \widehat{X_e}$. The resulting tree of coned-off spaces $P : \mathcal{TC}(X) \to T$ will be called the **induced tree of coned-off spaces**. The resulting space will thus be denoted as $\mathcal{TC}(X)$ when thought of as a tree of spaces. The **cone locus** of $\mathcal{TC}(X)$, is the graph (in fact a forest) whose vertex set $\mathcal{V}$ consists of the cone-points in the vertex set and whose edge-set $\mathcal{E}$ consists of the cone-points in the edge set. The incidence relations of the edges are same as that of the incidence relations in T. The cone locus is a forest as a single edge space cannot have more than one horosphere-like set mapping to a common horosphere-like set in a vertex-set.

Note that connected components of the cone-locus can be naturally identified with sub-trees of T. Each such connected component of the cone-locus will be called a **maximal cone-subtree**. The collection of *maximal cone-subtrees* will be denoted by $\mathcal{T}$ and elements of $\mathcal{T}$ will be denoted as T_α. Further, each maximal cone-subtree T_α naturally gives rise to a tree T_α of horosphere- like subsets Θ_α (depending on which cone-points arise as vertices and edges of T_α) as follows:
Let $x_v \in \mathcal{V}(T_\alpha)$, then x_v is a cone point over a unique horosphere-like set $H_{\alpha v}$ for some vertex space X_v and similarly for an edge $e = [w_1, w_2] \in \mathcal{E}(T_\alpha)$ there exists a unique horosphere-like set $H_{\alpha e}$ in some edge space X_e such that $f_{e,w_1}(H_{\alpha e} \times \{0\}) = H_{\alpha w_1}$ and $f_{e,w_2}(H_{\alpha e} \times \{1\}) = H_{\alpha w_2}$. Define

$$\Theta_\alpha := (\cup_{x_v \in \mathcal{V}(T_\alpha)} H_{\alpha v}) \bigcup (\cup_{e \in \mathcal{E}(T_\alpha)} f_e(H_{\alpha e} \times (0,1))).$$

Θ_α will be referred to as a **maximal cone-subtree of horosphere-like spaces**. $g_\alpha := P|_{\Theta_\alpha} : \Theta_\alpha \to T_\alpha$ will denote the induced tree of horosphere-like sets. $\mathcal{G}$ will denote the collection of these maps. The collection of Θ_α's will be denoted as $\mathcal{C}$.

Remark 2.17. *(1) If $P : X \to T$ is a tree of relatively hyperbolic spaces and $\mathcal{C}$ be the collection of maximal cone subtree of horosphere like spaces C_α, then the tree (T) of coned-off spaces $\mathcal{TC}(X)$, can be thought of as obtained from X by partial electrocuting each C_α to the cone subtree T_α.*
(2) If X is hyperbolic relative to the collection $\mathcal{C}$, then from lemma 2.13, $(\mathcal{TC}(X), d_{\mathcal{TC}(X)})$ is a hyperbolic metric space.

Thus we can treat the tree (T) of coned spaces as a partially electrocuted space $(X, \mathcal{C}, \mathcal{G}, \mathcal{T})$, where $\mathcal{G}$ is the collection of maps $g_\alpha \colon C_\alpha \to T_\alpha$ collapsing C_α, the tree of horosphere-like spaces to the underlying tree T_α.

2.5. **Hyperbolic Ladder and Retraction map.** Let $P : X \to T$ be a tree of relatively hyperbolic spaces. Given a geodesic segment $\widehat{\lambda} \subset \widehat{X}_{v_0}$ with end points lying outside horospheres-like sets, in [13] a quasiconvex set $B_{\hat{\lambda}} \subset \widehat{X}$, called hyperbolic ladder, is constructed. For the sake of completeness, we give the construction here. The following lemma will be used for the construction of hyperbolic ladder

Lemma 2.18. *(Lemma 3.1 of [10])*
Given $\delta > 0$, there exist D, C_1 such that if x, y are points of a $\delta-$ hyperbolic metric space (X, d), λ is a hyperbolic geodesic in X, and π_λ is a nearest point projection of X onto λ with $d(\pi_\lambda(x), \pi_\lambda(y)) \geq D$, then $[x, \pi_\lambda(x)] \cup [\pi_\lambda(x), \pi_\lambda(y)] \cup [\pi_\lambda(y), y]$ lies in a C_1-neighborhood of any geodesic joining x, y.

The quasi-isometric embeddings $f_{e,v} \colon X_e \to X_v$ induce quasi-isometric embeddings $f^h_{e,v} \colon X^h_e \to X^h_v$. Thus for every vertex v and edge e in T, $f^h_{e,v}(X^h_e)$ will be C_2 quasiconvex for some $C_2 > 0$. Let δ be the hyperbolicity constant of X^h_v, X^h_e. Let C_1 be as in Lemma 2.18. Let $C = C_1 + C_2$

For $Z \subset X^h_v$, let $N_C(Z)$ denote the C-neighborhood of Z in X^h_v, where C is as above.

Hyperbolic Ladder $B_{\hat{\lambda}}$

Recall that $P : \mathcal{TC}(X) \to T$ is the usual projection to the base tree.

For convenience of exposition, T shall be assumed to be rooted, i.e. equipped with a base vertex v_0. Let $v \neq v_0$ be a vertex of T. Let v_- be the penultimate vertex on the geodesic edge path from v_0 to v. Let e denote the directed edge from v_- to v.

Define $\phi_{v,e} \colon f_{e,v_-}(X_e \times \{0\}) \to f_{e,v}(X_e \times \{1\})$ as follows:
If $p \in f_{e,v_-}(X_e \times \{0\}) \subset X_{v_-}$, choose $x \in X_e$ such that $p = f_{e,v_-}(x \times \{0\})$ and define $\phi_{v,e}(p) = f_{e,v}(x \times \{1\})$.

Note that in the above definition, x is chosen from a set of bounded diameter.

Since $f_{e,v_-}|_{X_e\times\{0\}}$ and $f_{e,v}|_{X_e\times\{1\}}$ are quasi-isometric embeddings into their respective vertex spaces $\phi_{v,e}$'s are uniform quasi-isometries for all vertices. We shall denote $\mathcal{E}(X_v,\mathcal{H}_v) = \mathcal{E}(\mathcal{G}(X_v,\mathcal{H}_v),\mathcal{H}_v^h) = \widehat{X_v}$ and $\mathcal{E}(X_e,\mathcal{H}_e) = \mathcal{E}(\mathcal{G}(X_e,\mathcal{H}_e),\mathcal{H}_e^h) = \widehat{X_e}$.

Step 1

Let $\hat{\mu} \subset \widehat{X_v}$ be a geodesic segment in $(\widehat{X_v}, d_{\widehat{X_v}})$ with starting and ending points lying outside horoballs and μ be the corresponding electro-ambient quasi-geodesic in X_v^h (cf Lemma 2.11). Then $P(\hat{\mu}) = v$. For each edge e incident on v , but not lying on the geodesic (in T) from v_0 to v, consider the collection of edges $\{e\}$ for which $N_C^h(\mu)\cap f_{e,v}(X_e) \neq \emptyset$ and for each such e, choose $p_e, q_e \in N_C^h(\mu)\cap f_{e,v}(X_e)$ such that $d_{X_v^h}(p_e,q_e)$ is maximal. Let $\{v_i\}$ be the terminal vertices of edges e_i for which $d_{\widehat{X}_v}(p_{e_i},q_{e_i}) > D$, where D is as in Lemma 2.18 above. Let $\hat{\mu}_{v,e_i}$ be a geodesic in $\widehat{X}_v$ joining $\phi_{v,e_i}(p_{e_i})$ and $\phi_{v,e_i}(q_{e_i})$. Define

$$B^1(\hat{\mu}) = i_v(\hat{\mu}) \cup \bigcup_i \hat{\Phi}_{v,e_i}(\hat{\mu}_{v,e_i})$$

where $\hat{\Phi}_{v,e_i}(\hat{\mu}_{v,e_i})$ is an electric geodesic in $\widehat{X}_{v_i}$ joining $\phi_{v,e_i}(p_{e_i})$ and $\phi_{v,e_i}(q_{e_i})$.

Step 2

Step 1 above constructs $B^1(\hat{\lambda})$ in particular. We proceed inductively.

Suppose that $B^m(\hat{\lambda})$ has been constructed such that the vertices in of $P(B^m(\hat{\lambda})) \subset T$ are the vertices of a subtree. Let $\{w_i\}_i = P(B^m(\hat{\lambda})) \setminus P(B^{m-1}(\hat{\lambda}))$.

Assume further that $P^{-1}(w_k)\cap B^m(\hat{\lambda})$ is a path of the form $i_{w_k}(\hat{\lambda}_{w_k})$, where $\hat{\lambda}_{w_k}$ is a geodesic in $(\widehat{X}_{w_k}, d_{\widehat{X}_{w_k}})$.

Define

$$B^{m+1}(\hat{\lambda}) = B^m(\hat{\lambda}) \cup \bigcup_k (B^1(\hat{\lambda}_{w_k}))$$

where $B^1(\hat{\lambda}_{w_k})$ is defined in step 1 above.

Define

$$B_{\hat{\lambda}} = \cup_{m\geq 1} B^m(\hat{\lambda})$$

Observe that the vertices comprising $P(B_{\hat{\lambda}})$ in T are the vertices of a subtree, say, T_1.

Definition 2.19 (Retraction Map). *Let $\widehat{\pi}_{\widehat{\lambda}_v} : \widehat{X}_v \to \widehat{\lambda}_v$ be a nearest point projection of $\widehat{X}_v$ onto $\widehat{\lambda}_v$. $\widehat{\Pi}_{\widehat{\lambda}}$ is defined on $\bigcup_{v\in T_1} \widehat{X}_v$ by $\widehat{\Pi}_{\widehat{\lambda}}(x) = \widehat{i}_v(\widehat{\pi}_{\widehat{\lambda}_v}(x))$ for $x \in \widehat{X}_v$. If $x \in P^{-1}(T\setminus T_1)$, choose $x_1 \in P^{-1}(T_1)$ such that $d(x,x_1) = d(x,P^{-1}(T_1))$ and define $\widehat{\Pi}_{\widehat{\lambda}}(x) = x_1$. Next define $\widehat{\Pi}_{\widehat{\lambda}}(x) = \widehat{\Pi}_{\widehat{\lambda}}(\widehat{\Pi}'_{\widehat{\lambda}}(x))$.*

Theorem 2.20. [13] *There exists $C_0 \geq 0$ such that*

$$d_{\widehat{X}}(\widehat{\Pi}_{\widehat{\lambda}}(x), \widehat{\Pi}_{\widehat{\lambda}}(y)) \leq C_0 d_{\widehat{X}}(x,y) + C_0 \text{ for } x,y \in \widehat{X}.$$

In particular, if $\mathcal{TC}(X)$ is hyperbolic, then $B_{\widehat{\lambda}}$ is uniformly (independent of λ) quasiconvex.

2.6. **Quasigeodesic Rays.** Let $\widehat{\lambda_{v_0}}$ be an electric geodesic segment from a to b in $\widehat{X_{v_0}}$ with a and b lying outside horosphere-like sets. Let T_1 be subtree of T spanned by $P(B_{\widehat{\lambda}_{v_0}})$. Let $\lambda_v^b = \widehat{\lambda_v}\cap X_v$. (Note that $\lambda_v^b \subset \widehat{\lambda}_v$). Suppose $B_{\lambda_{v_0}^b} = \bigcup_{v\in T_1} i_v(\lambda_v^b)$. (Then $B_{\lambda_{v_0}^b} \subset B_{\widehat{\lambda}_{v_0}}$).

Lemma 2.21. [13] *There exists $C \geq 0$ such that for all $x \in \lambda_v^b \subset B^b_{\lambda_{v_0}}$ there exists a C- quasigeodesic map $r_x : S \to B^b_{\lambda_{v_0}}$ such that $r_x(v) = x$ and $d_S(v,w) \leq d(r_x(v), r_x(w) \leq C d_S(v,w)$, where S is the geodesic edge path in T_1 joining v and v_0 and $w \in S$.*

Remark 2.22. *For a tree $P : X \to T$ of relatively hyperbolic metric spaces and an electric geodesic segment $\widehat{\lambda_{v_0}}$, it follows from Lemma 2.21 that $B^b_{\lambda_{v_0}}$ lies in an C diam$(P(B^b_{\lambda_{v_0}}))$ neighborhood of $\lambda^b_{v_0}$.*

Definition 2.23. *An C-quasi-isometric section of $[v_0,v]$ ending at $x \in P^{-1}(v)\cap B(\lambda^b_{v_0})$ is the image of quasigeodesic map $r_x|_{[v_0,v]}$.*

3. Height

Definition 3.1. *Suppose G is hyperbolic relative to $\{H_1, ..., H_r\}$ and K is a relatively hyperbolic quasiconvex subgroup of G. The height of K is defined to be $(n-1)$ if n is the smallest number with the property that for any distinct left cosets $g_1K, ..., g_nK$, the intersection $\cap_{1\leq i\leq n} g_iKg_i^{-1}$ is elliptic or parabolic.*

Theorem 3.2. [7] *Let G be a relatively hyperbolic group, then height of a relatively quasiconvex group in G is finite.*

Definition 3.3. *([11])(Vertex Hallway) Let $P : X \to T$ be a tree of geodesic spaces and let $\rho > 0$. An ρ-thin hallway Θ is a collection of geodesics $\mu_i \subset X_{v_i}$, where $i \in 0, ..., n$ such that*

1. *$v_0, ..., v_n$ are successive vertices on a geodesic $[v_0, v_n]$ in T,*
2. *end points of μ_i are trapped by two ρ-quasi-isometric sections Σ_1 and Σ_2 i.e. if $\mu_i = [a_i, b_i]$ then $a_i = \Sigma_1(v_i)$ and $b_i = \Sigma_2(v_i)$.*

Length of the Hallway Θ is defined to be n.

Let $P : X \to T$ be a tree of relatively hyperbolic metric spaces with X being hyperbolic relative to the collection of maximal cone-subtree of horosphere-like spaces $\mathcal{C}$. For a geodesic subspace Y of a vertex space X_{v_0}, let $\widehat{Y}$ denote the coning-off Y in $\widehat{X}_{v_0}$. The following lemma says that, if $\widehat{Y}$ is not a quasiconvex subset in the tree of coned-off spaces $\mathcal{TC}(X)$, then there exists a sequence of hallways Θ_i such that length of Θ_i tends to infinity as $i \to \infty$. This fact, for the tree of hyperbolic metric spaces, was proved by Mitra in [11] (viz. Lemma 4.2 and Corollary 4.3 of [11]). The same argument goes through in the set up of relative hyperbolicity.

Lemma 3.4. *(Corollary 4.3 of [11])(Existence of Hallways) Suppose $P : X \to T$ is a tree of relatively hyperbolic spaces as in Definition 2.15 with X being hyperbolic relative to the collection of maximal cone-subtree of horosphere-like spaces $\mathcal{C}$. Let Y be a subspace of a vertex space X_{v_0} such that $\widehat{Y}$ is not quasiconvex in $\mathcal{TC}(X)$ then there exist relative geodesics $\mu_i \subset Y$ and ρ-thin hallways Θ_i in $\mathcal{TC}(X)$, with one end as μ_i, trapped by quasi-isometric sections $\Sigma_1, \Sigma_2 : P(\Theta_i) \to X$ such that the lengths of $\widehat{\mu}_i$ in $\widehat{Y}$ and the hallway Θ_i are greater than i.*

Remark 3.5. *Observe that $\mathcal{E}(X, \mathcal{C}) = \mathcal{E}(\mathcal{TC}(X), \mathcal{T})$. Therefore, if a set is quasiconvex in $\mathcal{TC}(X)$ then it is quasiconvex in $\mathcal{E}(X, \mathcal{C})$. Hence, in Lemma 3.4, we can take Y not to be quasiconvex in $\mathcal{E}(X, \mathcal{C})$.*

4. Main Theorem

Consider a graph of groups $(\mathcal{G}, \Lambda)$. For a vertex v (resp. edge e) of Λ, let G_v (resp. G_e) denote the vertex group (resp. edge group). The fundamental group $\pi_1(\mathcal{G}, \Lambda)$ of $(\mathcal{G}, \Lambda)$ is generated by the groups G_v $(v \in Vert(\Lambda))$ and the elements g_y $(y \in Edge(\Lambda))$. A graph of groups $(\mathcal{H}, \Gamma)$ is said to be **graph of subgroups** of $(\mathcal{G}, \Lambda)$ if $\Gamma = \Lambda$, H_v is subgroup of G_v for each vertex $v \in \Lambda$, and $\pi_1(\mathcal{H}, \Lambda)$ is also subgroup of $\pi_1(\mathcal{G}, \Lambda)$.

Theorem 4.1. *Let $\delta \geq 0, k \geq 1, \epsilon \geq 0$. Let $(\mathcal{G}, \Lambda)$ be a finite graph of groups with its fundamental group $G = \pi_1(\mathcal{G}, \Lambda)$ finitely generated and torsion free. Consider the graph of subgroups $(\mathcal{H}, \Lambda)$ of $(\mathcal{G}, \Lambda)$ with $H = \pi_1(\mathcal{H}, \Lambda)$ and suppose the following conditions hold:*
(i) for every edge e incident on a vertex v of the graph Λ, the embeddings $i_e : G_e \hookrightarrow G_v$, $\widehat{i}_e : \mathcal{E}(G_e, H_e) \hookrightarrow \mathcal{E}(G_v, H_v)$ are (k, ϵ)-quasi-isometric, where $H_e = H_v \cap G_e$,
(ii) G is δ-hyperbolic relative to H,
(iii) for each vertex v of Λ; G_v is δ-hyperbolic relative to H_v and the vertex group G_v has finite height in G.
Then, for all vertex w, G_w is relatively quasiconvex subgroup of G.

Proof. If possible, suppose G_{v_0} is not relatively quasiconvex in G. We will prove that there exists a vertex group G_v and an infinite sequence $\{n_i\}$ of natural numbers such that the following holds: for each n_i there exists $k = k(n_i)$-distinct cosets $g_{1k}G_v, ..., g_{kk}G_v$ such that $\bigcap_{r=1}^{k}(g_{rk}G_vg_{rk}^{-1})$ is hyperbolic and $k(n_i) \to \infty$ as $n_i \to \infty$.

Recall that, for the Bass-Serre covering tree $\widetilde{\Lambda}$, vertex set $Vert(\widetilde{\Lambda}) = \bigsqcup_{p\in Vert(\Lambda)} G/G_p$ and edge set $Edge(\widetilde{\Lambda}) = \bigsqcup_{e\in Edge(\Lambda)} G/G_e$. Two vertices $g_1G_{p_1}$ and $g_2G_{p_2}$ has an edge gG_e if $g_1G_{p_1} = gG_{p_1}$, $g_2G_{p_2} = gG_{p_2}$ and e is an edge between p_1, p_2 in Λ. The group G has a natural action on $\widetilde{\Lambda}$ with subgroup stabilising gG_p is gG_pg^{-1}. Associated with graph of groups $(\mathcal{G}, \Lambda)$ there is natural tree of spaces $P : \mathcal{Z} \to \widetilde{\Lambda}$ (as in Definition 2.15) as follows:
for each vertex $v = gG_p$ (resp. edge $f = gG_e$) of $\widetilde{\Lambda}$, define the vertex space Z_v (resp. edge space Z_f) to be $g\Gamma_{G_p}$ (resp. $g\Gamma_{G_e}$), where $\Gamma_{G_p}, \Gamma_{G_e}$ are Cayley graphs. The monomorphisms $G_e \hookrightarrow G_p$ induce natural embeddings of edge spaces into vertex spaces. $P : \mathcal{Z} \to \widetilde{\Lambda}$ is given by $P(z) = v$ if z lies in vertex space Z_v and if z lies in an edge space Z_f then $P(z)$ is defined to be mid-point of the edge f. By hypothesis, each local group is a relatively hyperbolic group, therefore by coning-off each local group we get a tree of coned-off spaces $P : \mathcal{TC}(\mathcal{Z}) \to \widetilde{\Lambda}$ whose vertex, edge spaces are of the form $g\mathcal{E}(G_p, H_p)$, $g\mathcal{E}(G_e, H_e)$ respectively.

Now, as G_{v_0} is not relatively quasiconvex, then for each $n \in \mathbb{N}$, there exists $x_n, y_n \in G_{v_0}$ such that a geodesic $\widehat{[x_n, y_n]}$ joining x_n, y_n in the coned-off space $\mathcal{E}(G, H)$ does not lie totally inside the n-neighborhood of G_{v_0} in $\mathcal{E}(G, H)$ i.e. $\widehat{[x_n, y_n]} \nsubseteq Nbhd_{\mathcal{E}(G,H)}(G_{v_0}; n)$. Let $\widehat{\lambda}_n$ be a geodesic in $\mathcal{E}(G_{v_0}; H_{v_0})$ joining x_n, y_n. Length of $\widehat{\lambda}_n$ tends to ∞ as $n \to \infty$, so we can assume length of each $\widehat{\lambda}_n$ is greater than 1. if $x_n^{-1}y_n$ lie in a parabolic subgroup $H_{v_0} \in \mathcal{H}_{v_0}$, then both x_n, y_n lie on the same left coset of H_{v_0}. Then distance in $\mathcal{E}(G_{v_0}, H_{v_0})$ between x_n, y_n is bounded above by 1. Thus, $x_n^{-1}y_n$ is not a parabolic element.

Recall from 2.5, the notion of hyperbolic ladder $B_{\widehat{\lambda}_n}$ in $\mathcal{TC}(\mathcal{Z})$ which is a quasiconvex set in $\mathcal{TC}(\mathcal{Z})$. Outside horosphere-like sets, $\widehat{[x_n, y_n]}$ lie in a bounded neighborhood of $B_{\widehat{\lambda}_n}$. Therefore, $diamP(B_{\widehat{\lambda}_n}) \to \infty$ as $n \to \infty$. Using Lemma 3.4, there exist ρ-thin Hallways Θ_n of length greater than n in $\mathcal{TC}(\mathcal{Z})$, with one end $\widehat{\lambda}_n$, trapped by quasi-isometric sections Σ_n^1, Σ_n^2. Note that, by Lemma 2.21, the image of quasi-isometric sections lie outside horosphere-like sets. As the hallway is ρ-thin, for any two successive vertices v, w, we have $d_{\mathcal{Z}}(\Sigma_n^r(v), \Sigma_n^r(w)) \leq \rho$ $(r = 1, 2)$. The metric $d_{\mathcal{Z}}$ on $\mathcal{Z}$ is same as the word metric on G. As G is finitely generated, there exists finitely many words of length at most ρ. By Pigeon hole principle, there exists an infinite sequence $\{n_i\}$ such that $d_{\mathcal{Z}}(\Sigma_{n_i}^1(v), \Sigma_{n_i}^1(w)) = d_{\mathcal{Z}}(\Sigma_{n_i}^2(v), \Sigma_{n_i}^2(w))$ for each successive vertices v, w in $P(B_{\widehat{\lambda}_{n_i}})$. Let $k = k(n_i)$ be the total number of vertices in the Hallway Θ_{n_i}. Let $v_{0k}, v_{1k}, ..., v_{kk}$ be successive vertices of the Hallway Θ_{n_i} and $\mu_{0k}, \mu_{1k}, ..., \mu_{kk}$ be successive relative geodesics in the respective vertex spaces with $\widehat{\mu_{0k}} = \widehat{\lambda_{n_i}}$. Note that $k(n_i) \to \infty$, as length of hallway Θ_{n_i} is greater than n_i. For $j > 1$, let $s_{jk} = (\Sigma_{n_i}^1(v_{jk}))^{-1}\Sigma_{n_i}^1(v_{(j-1)k}) = (\Sigma_{n_i}^2(v_{jk}))^{-1}\Sigma_{n_i}^2(v_{(j-1)k})$. Let w_{jk} be the element in the vertex group $X_{v_{jk}}$ representing the relative geodesic μ_{jk}. Then, for $j = 1, ..., k$

$$w_{0k} = (s_{1k}s_{2k}...s_{jk})w_{jk}(s_{1k}s_{2k}...s_{jk})^{-1}.$$

Each w_{jk} lie in a conjugate of some vertex group in the graph of groups $(\mathcal{G}, \Lambda)$. Consider the set $S_k = \{w_{0k}, ..., w_{kk}\}$, where $k = k(n_i) \to \infty$ as $n_i \to \infty$. As Λ is a finite graph, by Pigeon hole principle, there exists a vertex group G_v and an infinite subsequence $\{k_m(n_i)\}$ of $\{k(n_i)\}$ such that $Card[Conj(G_v) \cap S_{k_m}] \to \infty$ as $k_m \to \infty$, where $Conj(G_v) = \bigcup_{g\in G} gG_vg^{-1}$. Suppose $Conj(G_v) \cap S_{k_m} = \{w_{l_0k_m}, w_{l_2k_m}, ..., w_{l_rk_m}\}$, where $r = r(k_m) \to \infty$ as $k_m \to \infty$. Then, $w_{l_0k_m} = s_{l_1k_m}...s_{l_jk_m}w_{l_jk_m}(s_{l_1k_m}...s_{l_jk_m})^{-1}$. Let $x_{l_jk_m} = s_{l_1k_m}...s_{l_jk_m}$, then $w_{l_0k_m} = x_{l_jk_m}w_{l_jk_m}x_{l_jk_m}^{-1}$ for $j = 1, 2, ..., r$. Now, each $w_{l_jk_m}$ lie in a conjugate of G_v, therefore there exists $y_{l_jk_m} \in G$ such that $w_{l_jk_m} \in y_{l_jk_m}G_vy_{l_jk_m}^{-1}$. Thus, $w_{l_0k_m} \in \bigcap_{1\leq j\leq r}(x_{l_jk_m}y_{l_jk_m})G_v(x_{l_jk_m}y_{l_jk_m})^{-1}$. As $w_{l_0k_m}$ is a hyperbolic element, we have $\bigcap_{1\leq j\leq r}(x_{l_jk_m}y_{l_jk_m})G_v(x_{l_jk_m}y_{l_jk_m})^{-1}$ contains hyperbolic element. Let $g_{jm} = x_{l_jk_m}y_{l_jk_m}$ then we have distinct left cosets $g_{1m}G_v, ..., g_{rm}G_v$ such that $\bigcap_{1\leq j\leq r} g_{jm}G_vg_{jm}^{-1}$ contains hyperbolic element and $r \to \infty$. So, height of G_v is infinite. This contradicts the hypothesis that height of each vertex group is finite.

□

If $G = A *_C B$, then G acts on a simplicial tree T (Bass-Serre Tree) cocompactly such that the vertex stabilizers are conjugates of subgroups A and B; edge stabilizers are conjugates of subgroup

C. Suppose C_1 is a subgroup of C and G, A, B, C are all hyperbolic relative to the subgroup C_1. If C is not relatively quasiconvex in G then there exists infinitely many uniformly thin Hallways whose relative geodesics lie in C and length of Hallways diverges to ∞. Since there is only one edge group, the argument of Theorem 4.1 goes through and we have the following corollary:

Corollary 4.2. *Let $G = A *_C B$ and C_1 be a subgroup of C. Suppose G, A, B, C are all hyperbolic relative to the subgroup C_1 and C is relatively quasiconvex in A, B Then, height of C is finite in G if and only if C is relatively quasiconvex in G.*

References

[1] M.Bestvina & M.Feighn *A Combination Theorem for Negatively Curved Groups Journal of Differential Geometry, 35 (1992), 85-101*

[2] B. H. Bowditch. *Relatively hyperbolic groups, Internat. J. Algebra and Computation. 22 (2012) 1250016, 66pp.*

[3] Benson Farb. *Relatively hyperbolic groups, Geom. Funct. Anal. 8, pages 810-840, 1998*

[4] Rita Gitik, Mahan Mitra, Eliyahu Rips & Michah Sageev. *Width of Subgroups Transactions of AMS, Volume 350, Number 1, January 1998, Pages 321-329*

[5] M. Gromov. *Essays in Group Theory, (ed Gersten) MSRI Publ., vol. 8, Springer Verlag, pages 75-263, 1985.*

[6] Daniel Groves & J.F.Manning *Dehn filling in relatively hyperbolic groups, Israel Journal of Mathematics, 168 (2008), 317-429*

[7] G Christopher Hruska & Daniel T. Wise, *Packing subgroups in relatively hyperbolic groups, Geom. Topol. 13 (2009) 1945-1988.*

[8] G Christopher Hruska, *Relative Hyperbolicity and Relative Quasiconvexity for countable groups, Algebr. Geom. Topol. 10 (2010) 1807-1856.*

[9] Ilya Kapovich, *The Combination Theorem and Quasiconvexity*, *Int. J. Algebra Comput., 11, 185 (2001)*

[10] Mahan Mitra, *Cannon-Thurston Maps for trees of hyperbolic Metric Spaces Journal of Differential Geometry, Volume 48, Number 1, 1998, Pages 135-164*

[11] Mahan Mitra, *Height in splitting of hyperbolic groups, Proc. Indian Acad. Sci. (Math. Sci.) Vol. 114, No. 1, February 2004, Pages 39-54.*

[12] Mahan Mj. and Lawrence Reeves, *A combination Theorem for Strong Relative Hyperbolicity, Geometry & Topology 12 (2008) Pages 1777-1798*

[13] Mahan Mj and Abhijit Pal, Relative Hyperbolicity, Trees of Spaces and Cannon-Thurston Maps, Geom. Dedicata, Volume 151, Issue 1 (2011), Pages 59-75

[14] Andrzej Szczepanski, *Relatively Hyperbolic Groups Michigan Math. J. Volume 45, Issue 3 (1998), 611-618.*

[15] Wen-Yuan Yang, *Limit Sets of Relatively Hyperbolic Groups Geometriae Dedicata, February 2012, Volume 156, Issue 1, pp 1-12*

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